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DIRICHLET AVERAGES OF GENERALIZED FOX-WRIGHT FUNCTION

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ABSTRACT

The objective of this paper is to investigate the Dirichlet averages of the generalized Fox-wright hypergeometric function introduced by Wright in (1935).[9,10] The authors deduce representations for the Dirichlet averages $R_k(\beta, \beta'; x, y)$ of the generalized Fox-Wright function with the fractional integrals in particular Riemann-Liouville integrals. Special cases of the established results associated with generalized Fox-wright functions have been discussed.

KEYWORDS: Fox-Wright function, Dirichlet averages and Riemann-Liouville integrals.

Mathematical Subject Classifications: 26A33 and 33C20.

INTRODUCTION

The Dirichlet average of a function is a certain kind integral average with respect to Dirichlet measure. The concept of Dirichlet average was introduced by Carlson in 1977. It is studied among others by Carlson[1,2,4], Zu Castell[5], Massopust and Forster[6], Neuman and Vanfleet[7] and others. A detailed and comprehensive account of various types of Dirichlet averages has been given by Carlson in his monography.[3] In the paper Dirichlet averages of the generalized Fox-wright due to wright[9,10] have been studied by the authors.

This paper is devoted to investigation of the generalized Fox- wright functions (also known as Fox-wright psi function or just wright function) is a generalization of the generalized hypergeometric function pFq(z) based on an idea of E. Maitland wright (1935).

$$ {}_p\Psi_q\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \dots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \dots & (b_q;B_q)\end{matrix}\middle| z\right) = \sum_{n=0}^{\infty}\frac{\Gamma(a_1+nA_1)\Gamma(a_2+nA_2)\ \dots\ \Gamma(a_p+nA_p)}{\Gamma(b_1+nB_1)\Gamma(b_2+nB_2)\ \dots\ \Gamma(b_q+nB_q)}\frac{z^n}{n!} \quad (1) $$

Where $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$ (equality only holds for appropriately bounded z). The Foxwright function is a special case of the Fox- H-function (Srivastava 1984).[12]

$$ {}_p\Psi_q\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \dots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \dots & (b_q;B_q)\end{matrix}\middle| z\right) = $$

$$ H_{p,q+1}^{1,p}\left(\begin{matrix}(1-a_1;A_1)(1-a_2;A_2) & \dots & (1-a_p;A_p)\\ (0,1)(1-b_1;B_1)(1-b_2;B_2) & \dots & (1-b_q;B_q)\end{matrix}\middle| -z\right) \quad (2) $$

It follows from (2) that generalized Mittag- Leffler function $E_{\alpha,\beta}^{\gamma}(z)$ can be represented in terms of the wright function as,

$$ E_{\alpha,\beta}^{\gamma}(z) = \frac{1}{\Gamma(\gamma)}{}_1\psi_1\left(\begin{matrix}(\gamma;1)\\(\beta;\alpha)\end{matrix}\middle| z\right) $$

$$ E_{\alpha,\beta}(z) = {}_1\psi_1\left(\begin{matrix}(1;1)\\(\beta;\alpha)\end{matrix}\middle| z\right) = H_{1,2}^{1,1}\left(\begin{matrix}(1-a_1;A_1)\\(0,1)(1-b_1;B_1)\end{matrix}\middle| -z\right) $$

Mathematical Preliminaries

We give below some definitions which are necessary in this paper.

Standard simplex in $R^n, n \geq 1$: We denote the standard simplex in $R^n, n \geq 1$ by

$$ E = E_n = (u_1, u_2, \dots u_n); u_1 \geq 0, u_2 \geq 0, \dots, u_n \geq 0 \text{ and } u_1 + u_2 + u_3 + u_n \leq 1\}. $$

Dirichlet Measures: Let b $\in C^k >; K \geq 2$ and let $E = E_{k-1}$ be the standard simplex in R^{k-1}. The complex measure μ_b defined by[1]

$$ d_{\mu_b}(u) = \frac{1}{B(b)} u_1{}^{b_1-1} u_2{}^{b_2-1} u_3{}^{b_3-1} \dots $$

$$ u_k{}^{b_{k-1}-1}(1-u_1, 1-u_2, \dots 1-u_{k-1})^{b_k-1} d_{u_1} d_{u_2} d_{u_3} \dots d_{u_{k-1}}. $$

$$ \text{Here } B(b) = B(b_1, b, \dots b_k) = \frac{\Gamma(b_1)\Gamma(b_2)\dots\Gamma(b_k)}{\Gamma(b_1+b_2+\dots b_k)} $$

$C > = \{z \epsilon c : z \neq 0\}$

Dirichlet average: let Ω be a convex set in C and let $z = (z_1 , z_2 , \ldots z_n) \epsilon \Omega^n$, $n \geq 2$, and letf be a measurable function on Ω. Define,

$$F(b;z) = \int_{E_{n-1}} f(u o z)\, d_{\mu_b}(u),\; k \in R.$$

Where $d_{\mu_b}(u)$ is a Dirichlet Measure.

$$B(b) = B(b_1 , b , \ldots b_n) = \frac{\Gamma(b_1)\Gamma(b_2)\ldots\Gamma(b_n)}{\Gamma(b_1+b_2+\cdots.b_n)}, \qquad R(b_j) > 0,\ j = 1,2,3\ldots,n$$

And $u o z = \sum_{j=1}^{n-1} u_j z_j + (1 - u_1 \ \ldots - u_{n-1}) z_n$.

For $n = 1$, $f(b;z) = f(z)$, for $n = 2$, we have

$$d_{\mu_{\beta,\beta'}}(u) = \frac{\Gamma(\beta+\beta')}{\Gamma(\beta)\Gamma(\beta')} u^{\beta-1}(1-u)^{\beta'-1} d(u).$$

Carlson[3] investigated the average for $f(z) = z^k$, $k \epsilon R$,

$$R_k(b;z) = \int_{E_{n-1}} (u o z)^k\, d_{\mu_b}(u),\; k \in R$$

And for n = 2, Carlson proved that[4]

$$R_k(\beta,\beta'; x,y) = \frac{1}{B(\beta,\beta')} \int_0^1 [ux + (1-u)y]^k\, u^{\beta-1}(1-u)^{\beta'-1} d(u),$$

Where $\beta, \beta' \epsilon C$, $\min [R(\beta), R(\beta')] > 0$, $x, y \epsilon R$.

Our paper is devoted to the study of the Dirichlet averages of the generalized Fox- Wright function (1) in the form,

$$pMq\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \ldots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \ldots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] = \int_{E_1} {}_p\Psi_q(u o z)\, d_{\mu_{\beta,\beta'}}(u) \qquad (3)$$

Where $R(\beta) > 0$, $R(\beta') > 0$; $x, y \epsilon R$ and $\beta, \beta' \epsilon C$.

Reimann-Liouville fractional integral of order $\alpha \epsilon C$, $R(\alpha) > 0$.[13]

$$(I_{a+}^{\alpha} f)x = \frac{1}{\Gamma(\alpha)} \int_a^x (x-t)^{\alpha-1} f(t)\, dt,\ (x > a, a \epsilon R) \qquad (4)$$

Representation of R_k and pMq in terms of Reimann-Liouville fractional integrals

In this section we deduced representations for the Dirichlet averages $R_k(\beta,\beta',x,y)$ and $pMq(\beta,\beta';x,y)$ with fractional integral operators.

Theorem: Let $\beta,\beta' \in C$, $R(\beta) > 0$, $R(\beta') > 0$, and x,y be real numbers such that $x > y$ and $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{p} A_j \geq 0$, and pMq and I_{a+}^{α} be given by (3) and (4) respectively. Then the Dirichlet average of the generalized Fox-wright functions is given by,

$$pMq\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \dots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \dots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] =$$

$$\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\left[I_{0+}^{\alpha}\ {}_p\Psi_q\left(\begin{matrix}(a_1;A_1),(a_2;A_2)\dots(a_p;A_p)\\ (b_1;B_1),(b_2;B_2)\dots(b_q;B_q)\end{matrix}\middle| z\right)\right]$$

Where $\beta,\beta' \in C$, $R(\beta) > 0$, $R(\beta') > 0$, x,y ϵ R and $1+\sum_{j=1}^{q} B_j - \sum_{j=1}^{q} A_j \geq 0$ (equality only holds for appropriately bounded z).

Proof: According to equations (1) and (2) we have,

$$pMq\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \dots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \dots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta';x,y)\right)\right] =$$

$$\frac{1}{B(\beta,\beta')}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^1[y+u(x-y)]^n\,u^{\beta-1}(1-u)^{\beta'-1}d(u).$$

$$pMq\left[\left(\begin{matrix}(a_1;A_1)(a_2;A_2) & \dots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \dots & (b_q;B_q)\end{matrix}\middle|(\beta,\beta';x,y)\right)\right]$$

$$=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta')\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^1[y+u(x-y)]^n\,u^{\beta-1}(1-u)^{\beta'-1}d(u).$$

$Put\ u(x-y) = t$ in above equation , we get

$$\text{pMq}\left[\begin{pmatrix}(a_1;A_1)(a_2;A_2) & \dots & (a_p;A_p)\\ (b_1;B_1)(b_2;B_2) & \dots & (b_q;B_q)\end{pmatrix}\middle|(\beta,\beta';x,y)\right]$$

$$=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta')\,\Gamma(\beta)}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^{x-y}[y+t)]^n\left\{\frac{t}{x-y}\right\}^{\beta-1}\left(1-\frac{t}{x-y}\right)^{\beta'-1}\frac{dt}{x-y}.$$

$$=\frac{(x-y)^{1-\beta-\beta'}}{B(\beta,\beta')}\sum_{n=0}^{\infty}\frac{\prod_{j=1}^{p}\Gamma(a_j+nA_j)}{\prod_{j=1}^{q}\Gamma(b_j+nB_j)}\frac{1}{n!}\int_0^{x-y}[y+t]^n\{t\}^{\beta-1}(x-y-t)^{\beta'-1}dt.$$

$$=\frac{(x-y)^{1-\beta-\beta'}}{B(\beta,\beta')}\int_0^{x-y}t^{\beta-1}\text{p}\psi\text{q}\left(\begin{matrix}(a_j;A_j)_{1,p}\\(b_j;B_j)_{1,q}\end{matrix}\middle|y+t\right)(x-y-t)^{\beta'-1}dt.$$

$$=\frac{(x-y)^{1-\beta-\beta'}}{B(\beta,\beta')}\int_0^{x-y}t^{\beta-1}\text{p}\psi\text{q}\left(\begin{matrix}(a_j;A_j)_{1,p}\\(b_j;B_j)_{1,q}\end{matrix}\middle|y+t\right)(x-y-t)^{\beta'-1}dt.$$

This proves the theorem.

Special Cases

In this section, we consider some particular cases of the above theorem by setting p = q =1 and $a=\gamma$, $A=1$, $b=\beta$ and $B=\alpha$, we get well known result reported in[8] as follows,

$$M_{\alpha,\delta}^{\gamma}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\int_0^{x-y}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(y+t)\,(x-y-t)^{\beta'-1}dt.$$

$$M_{\alpha,\delta}^{\gamma}(\beta,\beta';x,y)=\frac{\Gamma(\beta+\beta')}{\Gamma(\beta)(x-y)^{\beta+\beta'-1}}\{I_{0+}^{\alpha}t^{\beta-1}E_{\alpha,\beta}^{\gamma}(y+t)\}(x-y).$$

Further, by setting y = 0 in above equations, we get well- known result reported in [11] like as

$$M_{\alpha,\delta}^{\gamma}(\beta,\beta';x,0)=\frac{\Gamma(\beta+\beta')}{\Gamma(\gamma)\;\Gamma(\beta)}2\psi2\left[\begin{matrix}(\gamma,1), & (\beta,1)\\(\beta+\beta',1), & (\delta,1)\end{matrix}\;,x\right].$$

In particular, when $\beta+\beta'=\gamma$,

$$M_{\alpha,\delta}^{\gamma}(\beta,\gamma-\beta;x,0)=\frac{1}{\Gamma(\beta)}1\psi1\left[\begin{matrix}(\beta,1)\\(\delta,\alpha)\end{matrix}\middle|x\right].$$

CONCLUSION

The results proved in this paper give some contributions to the theory of the generalized Fox-Wright hypergeometric function, especially Dirichlet averages. The results proved in this paper appear to be new and likely to have useful applications to a wide range of problems of mathematics, statistics and physical sciences.

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